AD-A114 317 STANFORD UNIV CA DEPT OF MATHEMATICS F/G 20/4
CONVECTIVE FLOW WITH SUBCRITICAL INSTABILITY.(U)
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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER: 17902.12-MA
2. GOVT ACCESSION NO.: N/A AD-A114 317
3. RECIPIENT'S CATALOG NUMBER: N/A
4. TITLE (and Subtitle): Convective Flow with Subcritical Instability
5. TYPE OF REPORT & PERIOD COVERED: Reprint
6. PERFORMING ORG. REPORT NUMBER: N/A
7. AUTHOR(s): John C. Neu
8. CONTRACT OR GRANT NUMBER(s): DAAG29 81 K 0032
9. PERFORMING ORGANIZATION NAME AND ADDRESS: Stanford University, Stanford, CA 94305
10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS: N/A
11. CONTROLLING OFFICE NAME AND ADDRESS: U. S. Army Research Office, P. O. Box 12211, Research Triangle Park, NC 27709
12. REPORT DATE: Jan 82
13. NUMBER OF PAGES: 6
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)
15. SECURITY CLASS. (of this report): Unclassified
15a. DECLASSIFICATION/DOWNGRADING SCHEDULE
16. DISTRIBUTION STATEMENT (of this Report)

Submitted for announcement only.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)
18. SUPPLEMENTARY NOTES
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

ARO 17902.12-MA

Convective flow with subcritical instability

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(Received 4 April 1980; accepted 23 October 1981)

An asymptotic analysis of subcritical instability in double diffusive convection is presented. Using a modified perturbation method, a Landau equation that determines how the amplitude of the convection evolves in time is derived. From the Landau equation, it is found that in certain cases, stable finite amplitude convection can exist even when the rest state with no flow is locally stable. The perturbation analysis complements and unifies previous work which is primarily qualitative or numerical in character.

I. INTRODUCTION

There are fluid flows which are stable against infinitesimal perturbations, yet finite amplitude disturbances, once created, may persist and even grow. We state that such flows exhibit subcritical instability. Certain geophysical and astrophysical convection processes provide prime examples. Veronis[1] discovered subcritical instability in the process of double diffusive convection. Here, the basic thermal instability is modified by the presence of a solute which introduces a second bouyancy force in addition to the usual effect of thermal expansion. Previous researchers into this subject have proceeded along the lines of a simplified semi-quantitative description, or a detailed numerical simulation. In this paper, we present a quantitative asymptotic analysis of double diffusive convection. Certain results of previous analyses are consequences of the unified treatment presented here.

To pose the proper goals of the analysis, we consider the physical mechanism of subcritical instability in double diffusive convection. In a fluid with no solute, where purely thermal convection takes place, the existence of convection due to an adverse temperature gradient is determined by a single dimensionless parameter R called the Rayleigh number. R is proportional to $\Delta T/\nu$, where ΔT is the temperature difference across the fluid layer and ν is the viscosity. There is a critical Rayleigh number $R=R^*$, such that the static solution corresponding to heat transfer purely by thermal convection without fluid flow is stable for $R<R^*$ and unstable for $R>R^*$. For $R>R^*$, the stable state is steady convection. The amplitude A of the convection, as measured by the velocity at a fixed point in the flow, increases like $(R-R^*)^{1/2}$ for $0<R-R^*\ll 1$. Curve (i) in Fig. 1 is a plot of the amplitude as a function of Rayleigh number for purely thermal convection and is typical of systems that exhibit supercritical bifurcation.

We now consider the modifications due to the presence of a solute. We assume that a higher solute concentration is maintained on the bottom of the fluid layer than on the top, so that the solute appears as a stabilizing agent. Hence, the critical Rayleigh number increases to a value $R_s>R^*$. We further assume that the solute diffusivity is sufficiently small so that the mixing due to convective motions is not overcome by the effect of fixed boundary concentrations. Under these conditions, it may be possible for finite amplitude convection to exist at thermal Rayleigh numbers $R<R_s$, because the convective motion maintains a nearly uniform solute concentration in the interior of the roll, and the stabilizing solute gradient does not appear there. Curve (ii) in Fig. 1 shows the conjectured bifurcation diagram for double diffusive convection. The locus of stable steady states is indicated by the solid lines, and the locus of unstable steady states by hatched lines. According to this picture, there is stable, finite amplitude convection for subcritical Rayleigh numbers in an interval $R_m<R<R_s$. We state that the convective process exhibits subcritical bifurcation. The goal of the present work is to provide an asymptotic description of this phenomenon.

Huppert and Moore[2] discovered a stable branch of solutions that represent finite amplitude convection. These solutions were computed both numerically and by analytic approximation based on mean field equations. This branch is represented by the solid portion of curve (ii) in Fig. 1. In addition, they discovered a branch of unstable solutions emanating from a bifurcation point at $R=R_s, S=0$. This branch is represented by the hatched portion of curve (ii). In this paper, we verify the conjecture of Huppert and Moore that the stable and unstable branches of solutions connect each other in a continuous fashion to form a single branch of solutions. In the limiting case with small solute diffusivity and small solute gradients, we present a unified analysis which describes the stable and unstable branches, and their joining to form one continuous family of solutions.

In Sec. II, we give the mathematical formulation of the double diffusive convection problem. In Sec. III, we present the results of linearized stability theory for periodic rolls. In Sec. IV, we obtain a first qualitative description of the subcritical instability by means of an averaging procedure due to Stuart. His procedure, although qualitative, nevertheless gives valuable insight into the conditions under which subcritical instability occurs. In particular, it suggests the proper limiting case for an asymptotic analysis. In Sec. V, we begin a formal asymptotic analysis analogous to the work of Keller and Kogelman on the Bénard problem.[3] The eventual result of this analysis is a Landau equation that describes the slow temporal variation of spatially periodic rolls. The derivation of the Landau equation involves an elliptic boundary value problem for the solute concentration. The completion of the analysis does not require the solution of the elliptic problem, but rather a certain inner product of the solution. There is a very convenient variational formulation suggested by Keller[4] for computing such inner products. In Sec. VI, we

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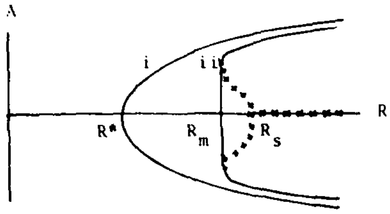

FIG. 1. Amplitude of convection A as a function of Rayleigh number R.

present the variational procedure and its application to our analysis. Finally, in Sec. VII we present the completed Landau equation and its predictions concern subcritical instability.

II. THE EQUATIONS OF MOTION

We consider a convective process taking place in a fluid layer of thickness d with both surfaces free. Gravity acts uniformly with magnitude g in the z direction. The density of the fluid depends on the temperature T and solute concentration S according to an equation of state

$$\rho = \rho_0[1 - \alpha_t(T - T_0) + \alpha_s(S - S_0)]\,. \tag{1}$$

Here, ρ_0 is the density when $T = T_0$ and $S = S_0$; α_t and α_s are both positive constants. Constant temperatures and solute concentrations are maintained on the free surfaces, with the higher values on the bottom surface. We denote the differences in temperature and solute concentration from bottom to top by ΔT and ΔS. In this analysis, we consider two-dimensional motions which depend only on x and z. We nondimensionalize by measuring all lengths in units of the fluid layer thickness d, and time in units of d^2/κ_t, where κ_t is the thermal diffusivity. Under these conditions, the dimensionless equations of the Boussinesq approximations are

$$(\partial_t - \sigma\nabla^2)\nabla^2\psi = -\sigma R\theta_x + \sigma S\Sigma_x - \psi_z\nabla^2\psi_x + \psi_x\nabla^2\psi_z\,, \tag{2a}$$

$$(\partial_t - \nabla^2)\theta + \psi_x = -\psi_z\theta_x + \psi_x\theta_z\,, \tag{2b}$$

$$(\partial_t - \tau\nabla^2)\Sigma + \psi_x = -\psi_z\Sigma_x + \psi_x\Sigma_z\,, \tag{2c}$$

for $0 < z < 1$. Here, ψ is the stream function from which one may determine the horizontal and vertical velocities according to $u = \psi_z$ and $w = -\psi_x$. θ and Σ are the deviations of temperature and solute concentration from the rest state with no fluid flow. ψ, θ, and Σ are the unknowns to be solved for. There are four dimensionless numbers: σ, the Prandtl number, is the ratio of viscous diffusivity ν to thermal diffusivity κ_t. τ is the ratio of solute diffusivity κ_s to thermal diffusivity. $R \equiv g\alpha_t\Delta T d^3/\kappa_t\nu$ and $S \equiv g\alpha_s\Delta S d^3/\kappa_t\nu$ are the thermal and solute Rayleigh numbers, respectively. Equation (2a) governs the momentum. The bouyancy force terms are $-\sigma R\theta_x$ and $\sigma S\Sigma_x$. Equations (2b) and (2c) express the conduction of heat and solute.

The boundary conditions at the free surfaces are

$$\psi = \psi_{zz} = 0, \quad \text{on } z = 0, 1\,. \tag{3a}$$

$$\theta = \Sigma = 0, \quad \text{on } z = 0, 1\,. \tag{3b}$$

The conditions (3a) imply $w = -\psi_x = 0$ and $u_z = \psi_{zz} = 0$. The condition $w = 0$ at $z = 0$ and $z = 1$ expresses the confinement of the fluid between the planes $z = 0$ and $z = 1$. $u_z = 0$ is the "free surface" condition of no tangential stress. The conditions (3b) follow from the choice of fixing the temperature and solute concentration at the boundaries.

The system (2a)–(2c), (3a), (3b) is the mathematical formulation of a double diffusive convective process. The problem is to solve Eqs. (2a)–(2c) for ψ, θ, and Σ subject to the boundary conditions (3a) and (3b).

III. LINEAR STABILITY THEORY

The rest state with no convective motion is characterized by ψ, θ, $\Sigma \equiv 0$. From a linearized analysis of the problem (2a)–(2c), (3a), (3b), we see that this rest state is unstable against spatially periodic perturbations of the wavenumber k if the thermal Rayleigh number R exceeds the critical value

$$R_s = R^* + S/\tau, \quad R^* \equiv (\pi^2 + k^2)^3/k^2\,. \tag{4}$$

R^* is the critical value for purely thermal convection with $S = 0$. If $S > 0$, then $R_s > R^*$. We see that the presence of a solute whose concentration increases with depth has a stabilizing effect. If $R = R_s$, we have the condition of neutral stability for which the linearized problem admits a time independent solution. This solution is given by

$$\psi = a_\psi \sin kx \sin \pi z, \quad \theta = a_\theta \cos kx \sin \pi z\,,$$

$$\Sigma = a_\Sigma \cos kx \sin \pi z\,, \tag{5}$$

where

$$a_\theta = -(k/\pi^2 + k^2)a_\psi, \quad a_\Sigma = -(1/\tau)(k/\pi^2 + k^2)a_\psi\,. \tag{6}$$

Figure 2 depicts the streamlines, which are curves of constant ψ.

IV. THE QUALITATIVE THEORY OF SUBCRITICAL INSTABILITY

Chandrasekhar[5] has analyzed finite amplitude thermal convection by an averaging procedure called Stuart's method. We generalize his work to the case of double diffusive convection. The essence of the procedure is simple. From the equations of motion (2a)–(2c), (3a), (3b), we derive three integral relations for steady convection which express the balances between the creation and removal of mechanical energy, heat and solute. These integral relations are given by

$$\int_0 \langle\psi_z\nabla^2\psi_z\rangle dz - R\int_0^1 \langle\psi_x\vartheta\rangle dz + S\int_0^1 \langle\psi_x s\rangle dz = 0\,, \tag{7a}$$

$$\int_0^1 \langle\vartheta\nabla^2\vartheta\rangle dz - \int_0^1 \langle\psi_x\vartheta\rangle dz = \int_0^1 \langle\psi_x\vartheta\rangle^2 dz - \left(\int_0^1 \langle\psi_x\vartheta\rangle dz\right)^2, \tag{7b}$$

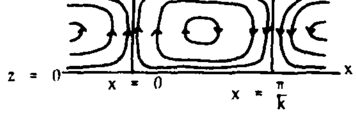

FIG. 2. Streamlines for neutrally stable convection.

$$\tau \int_0^1 \langle s \nabla^2 s \rangle dz - \int_0^1 \langle \psi_x s \rangle dz$$
$$= \frac{1}{\tau} \int_0^1 \langle \psi_x s \rangle^2 dz - \left(\int_0^1 \langle \psi_x s \rangle dz \right)^2 . \tag{7c}$$

Here, the brackets $\langle\ \rangle$ denote the average over a horizontal plane, while $\vartheta \equiv \theta - \langle \theta \rangle$ and $s \equiv \Sigma - \langle \Sigma \rangle$.

Near marginal stability where $0 < R - R_s \ll 1$, the convection is of small amplitude and the field variables ψ, θ, and Σ are well-approximated by the values (5) of the linearized theory. Upon substituting these values into the integral relations, we obtain the following system of equations for the amplitudes a_ψ, a_θ, and a_Σ:

$$(k^2 + \pi^2) a_\psi^2 + Rk a_\theta a_\psi - Sk a_\Sigma a_\psi = 0 , \tag{8a}$$

$$(k^2 + \pi^2) a_\theta^2 + k a_\theta a_\psi = -\tfrac{1}{8} k^2 a_\theta^2 a_\psi^2 , \tag{8b}$$

$$\tau (k^2 + \pi^2) a_\Sigma^2 + k a_\Sigma a_\psi = -(1/8\tau) k^2 a_\Sigma^2 a_\psi^2 . \tag{8c}$$

One solution of these equations is $a_\psi = a_\theta = a_\Sigma \equiv 0$, corresponding to the rest state with no convection; but there are nonzero solutions as well. By eliminating a_θ and a_Σ, we obtain the following equation relating the streamfunction amplitude a_ψ to the Rayleigh numbers R and S:

$$\frac{R - R^*}{\tau^2} = R^* a^2 + \frac{S}{\tau^3} \frac{1 + \tau^2 a^2}{1 + a^2} , \quad a \equiv k a_\psi [8(\pi^2 + k^2)\tau]^{1/2} . \tag{9}$$

Figure 3 is a graph based on Eq. (9) of the amplitude a as a function of $(R - R^*)/\tau^2$ for various values of S and τ.

If $S \geq 0$ and

$$\Delta \equiv (1/\tau^3 - 1/\tau) S - R^* < 0 , \tag{10}$$

then (9) yields supercritical bifurcation represented by curve (i). If $\Delta > 0$, there is subcritical bifurcation as represented by curve (iii). Curve (ii) represents the intermediate case $\Delta = 0$, in which the bifurcation curve makes fourth-order contact with its vertical tangent at the point P.

V. PERTURBATION ANALYSIS

We present a quantitative, asymptotic analysis of double diffusive convection to compliment previous works in the subject which have been either qualitative or numerical in character. The asymptotic limit to be considered is $S \to 0$. We take $S = \epsilon$, $0 < \epsilon \ll 1$. From the qualitative theory in Sec. IV, we expect subcritical bifurcation to occur when $\Delta \equiv (1/\tau^3 - 1/\tau) S - R^* > 0$. Since S is of order ϵ, we see that subcritical bifurcation occurs if τ, the ratio of solute to thermal diffusivities, is of order $\epsilon^{1/3}$. We set $\tau = \epsilon^{1/3} \tau_0$, where τ_0 is of order unity. Given that S and τ have magnitudes ϵ and $\epsilon^{1/3}$, the proper scaling of the remaining parameters and variables is determined from the following considerations.

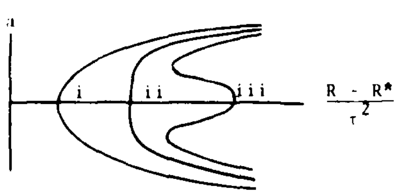

FIG. 3. Bifurcation diagrams based on Stuart's method.

(i) The fluxes of solute due to diffusive and convective effects should balance. In the solute conduction equation (2c) the term $\tau \nabla^2 \Sigma$ represents the effect of diffusion, and the terms $-\psi_z \Sigma_x + \psi_x \Sigma_z$ represent the effect of convection. Since τ is $O(\epsilon^{1/3})$, we see that ψ is $O(\epsilon^{1/3})$.

(ii) For purely thermal convection with $0 < R - R^* \ll 1$, the amplitudes of ψ and θ are both proportional to $(R - R^*)^{1/2}$. Although this relation is not valid quantitatively for double diffusive convection, we assume that it indicates correct orders of magnitude. Hence, θ is also $O(\epsilon^{1/3})$ and $R - R^*$ is $O(\epsilon^{2/3})$. For Rayleigh numbers R with $R - R^* = O(\epsilon^{2/3})$, the growth rate of temporal instability is $O(\epsilon^{2/3})$. Hence, the time scale for the growth of convection is $O(\epsilon^{-2/3})$.

(iii) The magnitude of the thermal bouyancy force due to thermal expansion is $R\theta$. If $R > R^*$, a solute free fluid experiences convective instability. Hence, we may think of $R^*\theta$ as the portion of the thermal bouyancy force that is required to overcome the stabilizing effect of viscosity, while $(R - R^*)\theta$ represents the actual motive force of the instability. In the case we consider, where a solute is present, we expect that this destabilizing force will be balanced in magnitude by the solute bouyancy force $S\Sigma$. Since $(R - R^*)\theta$ is $O[(\epsilon^{2/3})(\epsilon^{1/3})] = O(\epsilon)$ and $S = \epsilon$, we find that Σ is $O(1)$.

On the basis of the discussion in (i), (ii), and (iii), we adopt the following scaling of the variables:

$$\psi = \epsilon^{1/3} \Psi , \quad \theta = \epsilon^{1/3} \Theta , \quad \Sigma = \xi , \tag{11}$$
$$R = R^* + \epsilon^{2/3} r , \quad t = \epsilon^{2/3} T .$$

We seek asymptotic solutions for Ψ, Θ, and ξ in the form

$$\Psi \sim \Psi^0 + \epsilon^{1/3} \Psi^1 + \epsilon^{2/3} \Psi^2 + \ldots ,$$
$$\Theta \sim \Theta^0 + \epsilon^{1/3} \Theta^1 + \epsilon^{2/3} \Theta^2 + \ldots , \tag{12}$$
$$\xi \sim \xi^0 + \epsilon^{1/3} \xi^1 + \epsilon^{2/3} \xi^2 + \ldots .$$

We discuss the leading order solution. From Eqs. (2a) and (2b) we find that Ψ^0 and Θ^0 satisfy

$$\nabla^2 \Psi^0 - R^* \Theta^0_x = 0 , \quad \nabla^2 \Theta^0 - \Psi^0_x = 0 . \tag{13}$$

The solutions which satisfy the boundary conditions $\Psi^0 = \Psi^0_{zz} = 0$, $\Theta^0 = 0$ at $z = 0$ and $z = 1$ are

$$\Psi^0 = \alpha \sin kx \sin \pi z , \quad \Theta^0 = -[k/(k^2 + \pi^2)] \alpha \cos kx \sin \pi z . \tag{14}$$

Here, $\alpha = \alpha(T)$ represents the time varying amplitude of convection. The goal of this analysis is to find its governing equation.

From the solute conduction Eq. (2c) we find that ξ^0, satisfies

$$\frac{\tau_0}{\alpha} \nabla^2 \xi^0 - \Psi^0_z \xi^0_x + \Psi^0_x \xi^0_z = \Psi^0_x . \tag{15}$$

If we set $\Psi^0=\alpha\sin kx\sin\pi z$, this becomes

$$(\tau_0/\alpha)\nabla^2\xi^0-\phi_z\xi^0_x+\phi_x\xi^0_z=\phi_x,\quad \phi=\sin kx\sin\pi z\ . \quad (16)$$

Due to the periodicity of ϕ in the x direction, it is sufficient to solve for ξ^0 inside the single convection cell depicted in Fig. 3 with $0\le z\le 1$ and $0\le x\le\pi/k$. The boundary condition on $z=0$ and $z=1$ is $\xi^0=0$. Along the interfaces of the convection cell with its neighbors, symmetry dictates $\xi^0_x=0$.

To complete the leading order description, we need the evolution equation of the amplitude $\alpha(T)$. This is found in the process of solving the higher order equations. We solve the equations for the first order corrections Ψ^1 and Θ^1 and substitute the results into the equations for the second order corrections Ψ^2 and Θ^2. The second order equations have a solvability condition which yields the governing Landau equation for $\alpha(T)$. The analysis is straightforward and gives the result

$$\frac{1+\sigma}{\sigma}(k^2+\pi^2)\alpha_T=\frac{rk^2}{k^2+\pi^2}\alpha-\frac{1}{8}k^2(k^2+\pi^2)\alpha^3+(\xi^0,\phi_x)\ , \quad (17)$$

where (ξ^0,ϕ_x) is

$$(\xi^0,\phi_x)=\frac{4k}{\pi}\int_0^1\int_0^{\pi/k}\xi^0\phi_x\,dx\,dz\ . \quad (18)$$

The first two terms on the right-hand side of (17) appear in the usual Landau equation for purely thermal convection. The term (ξ^0,ϕ_x) represents the effect of solute. It is a definite function of the amplitude α because α appears as a parameter in Eq. (15). A variational principle suggested by Keller[4] provides a very convenient tool for estimating the functional dependence of (ξ^0,ϕ_x) on α.

VI. A VARIATIONAL PRINCIPLE AND ITS APPLICATION

Let u,a,b be functions defined in a bounded domain. We wish to compute the value of the inner product (u,b) when u is the solution of the inhomogeneous elliptic equation

$$Lu=a\ , \quad (19)$$

subject to homogeneous boundary conditions. This problem has a simple variational formulation: Let u^* be the solution of (19) and let v^* be the solution of the adjoint problem

$$L^\dagger v=b\ . \quad (20)$$

We define a functional $g(u,v)$ by

$$g(u,v)=(a,v)+(u,b)-(Lu,v)\ . \quad (21)$$

A simple calculation shows that

$$g(u^*+\sigma_1,v^*+\sigma_2)=(u^*,b)-(L\sigma_1,\sigma_2)\ . \quad (22)$$

We see that g attains its stationary value when $u=u^*$ and $v=v^*$, and that the stationary value is precisely the required inner product (u^*,b). In the self-adjoint case, the stationary value is also extremal. Since the problem we consider is not self-adjoint, the stationary value is not necessarily extremal.

The application to the analysis of Sec. V is clear. We apply variational principle with

$$L=(\tau_0/\alpha)\nabla^2-\phi_z\partial_x+\phi_x\partial_z,\quad a=b=\phi_x\ . \quad (23)$$

The functional $g(u,v)$ is

$$g(u,v)=(\phi_x,v)+(u,\phi_x)-[(\tau/2)\nabla^2u-\phi_zu_x+\phi_xu_z,v]\ , \quad (24)$$

where the inner product is the one defined by (18).

To compute the stationary value of g, it appears that one must perform variations with respect to both u and v; but, there is a symmetry in our particular problem which allows a simplification. The adjoint operator is

$$L^\dagger=(\tau_0/\alpha)\nabla^2+\partial_x\phi_z-\partial_z\phi_x=(\tau_0/\alpha)\nabla^2+\phi_z\partial_x-\phi_x\partial_z\ . \quad (25)$$

Hence, the adjoint equation $L^\dagger v=b$ is

$$(\tau_0/\alpha)\nabla^2v+\phi_zv_x-\phi_xv_z=\phi_x\ . \quad (26)$$

The function $\phi(x,z)=\sin kx\sin\pi z$ has the property $\phi(x,z)=\phi(x,1-z)$. As a consequence of the symmetry, we find that the adjoint Eq. (26) becomes equivalent to the original equation $Lu=a$ under the change of variable $z\to\bar z=1-z$. Hence, the solution v^* of the adjoint problem is obtained by substituting $1-z$ for z in $u^*(x,z)$; that is,

$$v^*(x,z)=u^*(x,1-z)\ . \quad (27)$$

As a result, the stationary value of $g(u,v)$ is equal to the stationary value of

$$\bar g(u)\equiv g[u(x,z),u(x,1-z)]\ . \quad (28)$$

To estimate the stationary value of $\bar g(u)$, we substitute into (27) a simple approximation to u with undetermined parameters and then compute the stationary value with respect to the parameters. To third order in α/τ_0, the solution of

$$Lu\equiv(\tau_0/\alpha)\nabla^2u-\phi_zu_x+\phi_xu_z=\phi_x\equiv a\ , \quad (29)$$

subject to boundary conditions

$$u=0 \text{ on } z=0,1,\quad u_x=0 \text{ on } x=0,\pi/k\ , \quad (30)$$

has the form

$$u=c_1\cos kx\sin\pi z+c_2\sin 2\pi z+c_3\cos kx\sin 3\pi z\ . \quad (31)$$

This is a natural choice for u. We substitute (31) into the expression (28) for $\bar g$. The stationary value of $\bar g$ with respect to the parameters c_1, c_2, and c_3 is

$$g^*=-\sqrt{8}k(k^2+\pi^2)^{1/2}A\left[1+3A^2+2\left(\frac{k^2+5\pi^2}{k^2+9\pi^2}\right)A^4\right]\times(1+2A^2)^{-2}\ , \quad (32)$$

where

$$A\equiv\frac{k}{[3(k^2+\pi^2)]^{1/2}}\frac{\alpha}{\tau_0}\ . \quad (33)$$

We take g^* as our approximation to the inner product (ξ^0,ϕ_x).

The accuracy of the approximation is easily accessed. The variational procedure generates values of c_1,c_2, c_3, so that the resulting expression (31) for u is correct

to third order in A. That is,

$$u = u^* + \sigma, \tag{34}$$

where u^* is the exact solution, and σ is $O(A^4)$. From (22) we find for this value of u that

$$\begin{aligned} g^* = \bar{g}(u) &= g[u(x,z), u(x,1-z)] \\ &= (u^*, b) - [\sigma(x,z), \sigma(x,1-z)]. \end{aligned} \tag{35}$$

Since the operator L in (29) is $O(1/A)$ and σ is $O(A^4)$, we find that the error term $[L\sigma(x,z), \sigma(x,1-z)]$ is $O(A^7)$. Hence, the approximation of (Σ^0, ϕ_x) in (32) has error $O(A^7)$.

VII. THE LANDAU EQUATION AND ITS CONSEQUENCES

Using the estimate of (ξ^0, ϕ_x) determined in Sec. VI, the Landau Eq. (17) can be written as

$$\lambda A_T = r'A - A^3 - \mu A\left[1 + 3A^2 + 2\left(\frac{k^2+5\pi^2}{k^2+9\pi^2}\right)A^4\right] \times (1+2A^2)^{-2} + O(A^7), \tag{36}$$

where $\lambda \equiv (1+\sigma)/[\sigma\tau_0^2(k^2+\pi^2)]$, $r' \equiv r/(R^*\tau_0^2)$ and $\mu \equiv 1/R^*\tau_0^3$. Steady rolls have $A_T = 0$. Their amplitudes satisfy

$$r' = A^2 + \mu\left[1 + 3A^2 + 2\left(\frac{k^2+5\pi^2}{k^2+9\pi^2}\right)A^4\right] \times (1+2A^2)^{-2} + O(A^6). \tag{37}$$

Figure 4 depicts A as a function of r' for various values of μ and $k = \pi\sqrt{2}$. The curves with $\mu > 1$ correspond to subcritical bifurcation. On any given curve, the portions with $dr'/dA^2 > 0$ correspond to stable steady states, while the portions with $dr'/dA^2 < 0$ correspond to unstable steady states. The unstable portions are indicated in Fig. 4 by hatched curves.

We consider the asymptotic validity of the bifurcation diagrams. We expand (37) as

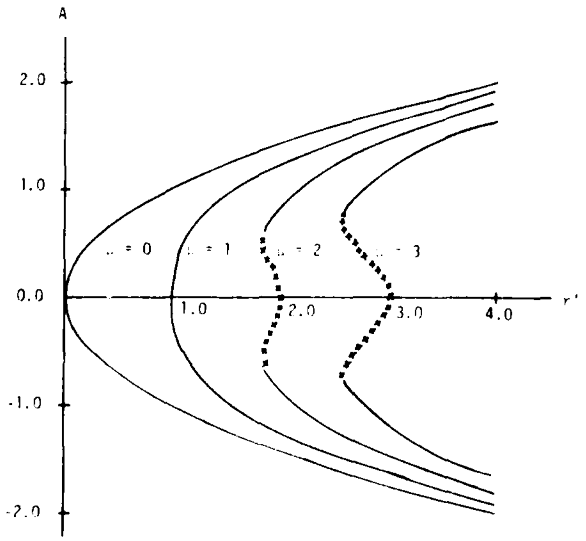

FIG. 4. Bifurcation diagrams based on variational method.

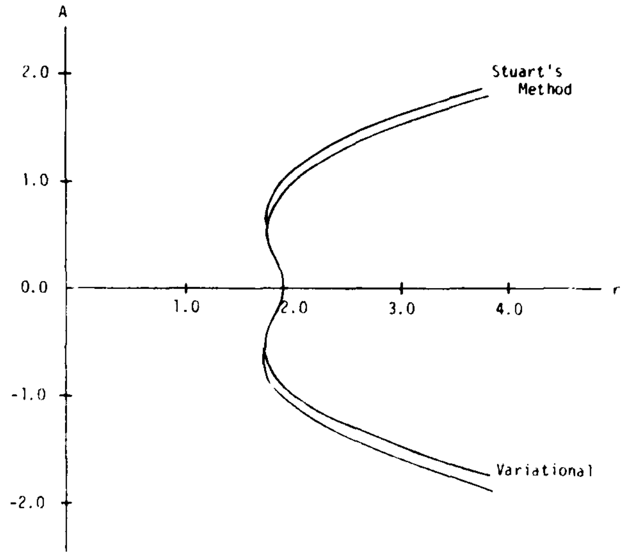

FIG. 5. Comparison between Stuart's method and variational procedure.

$$r' = \mu - (\mu-1)A^2 + 2\mu[(k^2+5\pi^2)/(k^2+9\pi^2)]A^4 + O(A^6). \tag{38}$$

The characteristic shape of the subcritical bifurcation curves in Fig. 4 is due to the balance of the quadratic and quartic terms in (38). Hence, A^2 is $O(\mu-1)^2$. The error $O(A^6)$ will be negligible to leading order if $(\mu-1)^4 \gg (\mu-1)^6$, or $\mu - 1 \ll 1$.

We compare the result (37) of the variational analysis with the result (9) obtained by Stuart's method. Setting $R = R^* + \epsilon^{2/3}r = R^*(1+\epsilon^{2/3}\tau_0^2 r')$, $a = A$, $S = \epsilon$, and $\epsilon = \epsilon^{1/3}\tau_0$, we find that (9) becomes

$$r' = A^2 + \mu/(1+A^2) + O(\epsilon^{2/3}). \tag{39}$$

Figure 5 shows graphs of A vs r' for $\mu = 2$ as obtained by Stuart's method and the variational procedure.

VIII. COMPARISON WITH RELATED WORK

Most numerical solutions of the double diffusive convection problem are performed at values of the thermal and saline Rayleigh numbers for which the flow is fully nonlinear. In these cases, the perturbation analysis does not apply. In the numberical work of Huppert and Moore,[2] typical values of $\mu \equiv S/R^*\tau^3$ range between 10 and 1,000, whereas the results of the perturbation analysis presented in Sec. VII are valid for $|\mu - 1| \ll 1$. It seems that the perturbation method, which gives accurate analysis locally, does not provide global results. In this respect, certain semi-quantitative methods perform better by providing crude, but globally applicable results.

Veronis[1] provided the first semi-quantitative analysis of subcritical convection through the use of a truncated modal representation. The result of his analysis is a fifth-order system of ordinary differential equations for the amplitudes of the fundamental modes and second-harmonic corrections. From these equations, he de-

TABLE I. Critical thermal Rayleigh number for finite amplitude convection as a function of saline Rayleigh number.

S	Numerical	r'_{min} Formula (41)	Formula (42)
10^3	20.4	18.6	16.9
$10^{3/2}$	52.3	39.9	37.6
10^4	148	91.9	89.7
2.5×10^4	358	189	185

termines the minimum value of the Rayleigh number R for which there is steady finite amplitude convection. His result is

$$R_{min} = (\sqrt{\tau S} + \sqrt{R^*})^2 , \tag{40}$$

or

$$r'_{min} = 2\mu^{1/2} + \tau^2\mu , \tag{41}$$

where $\mu \equiv S/R^*\tau^3$ and $r'_{min} = (R_{min} - R^*)/\tau^2$. The value of r'_{min} that we compute on the basis of Stuart's method is

$$r'_{min} = 2\mu^{1/2}(1-\tau^2)^{1/2} + \mu\tau^2 - 1 . \tag{42}$$

In the asymptotic analysis of this study, $\mu - 1$ and τ are small. In the limit $\mu \to 1, \tau \to 0$, Veronis' formula (41) predicts $r'_{min} = 2$, while formula (42) gives $r'_{min} = 1$. The latter value is the asymptotically correct one.

We compare the predictions (41) and (42) of the semiquantitative theories with the numerical work of Huppert and Moore. From their solutions, they estimate R'_{min} as a function of S for fixed values of τ. Table I compares the numerical and semiquantitative results in the case $\tau = 10^{-1/2}$.

ACKNOWLEDGMENTS

Research supported in part by the National Science Foundation, the Army Research Office, the Air Force Office of Scientific Research, and the Office of Naval Research.

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